

Black hole state counting in loop quantum gravity

A. Ghosh^{*} and P. Mitra[†]

Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064

The two ways of counting microscopic states of black holes in the U(1) formulation of loop quantum gravity, one counting all allowed spin network labels j, m and the other only m labels, are discussed in some detail. The constraints on m are clarified and the map between the flux quantum numbers and m discussed. Configurations with $|m| = j$, which are sometimes sought after, are shown to be important only when large areas are involved. The discussion is extended to the SU(2) formulation.

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^{*} amit.ghosh@saha.ac.in

[†] parthasarathi.mitra@saha.ac.in

Introduction: Loop quantum gravity has yielded a detailed prescription for identifying microscopic quantum states corresponding to an isolated horizon [1–3]. The horizon quantum states arise when the cross sections of the horizon are punctured by spin networks that live in the bulk. It can be shown that the spin quantum numbers j, m , which characterize the spin network, can also be used to label the quantum states of the horizon. The states counted are the ones that are consistent with a fixed area of the cross section and the boundary conditions imposed on the horizon. An improved estimation of the number of states was carried out in [4] counting only configurations with distinct m -labels – see also [5]. In an alternative scheme [6], the j -labels are also recognized as characterizing black hole microstates. In counting m -labels as in [4], an additional $|m| = j$ projection may come into play [7]. In this work we propose to elucidate all restrictions on these m quantum numbers and the $|m| = j$ prescription. It should be clarified that although the horizon area is taken to be more or less fixed, the complications which arise from fixing it with precision [8] are avoided here.

It may be noted that the above calculations were done in the framework of a $U(1)$ Chern-Simons theory of isolated horizons. Instead, the counting of states can also be carried out in an unbroken $SU(2)$ Chern-Simons theory (*cf.* [9]), an isolated horizon formulation for which is possible [10]. We shall comment on the differences which arise in this formulation.

In the m -counting scheme – originally proposed by [3] and used in [4, 5] – one counts only surface states labelled by the flux quantum numbers associated with punctures on the horizon. There is a map between the flux quantum numbers and m , so that one counts only those m -states that obey an area bound. On the other hand, j quantum numbers label area eigenstates, which belong to the *bulk* Hilbert space. In m -counting, states carrying different j quantum numbers but the same m quantum number are considered equivalent [5, 7], while in (j, m) -counting, such states are *not* identified. In (j, m) -counting the implementation of the area constraint is rather straightforward, while in m -counting there can only be a bound $|m| \leq j$, so that the area constraint has to be implemented only through some inequality. In the following we show that a one-to-one map from the flux quantum numbers to m exists for both counting schemes.

Restrictions on m : We start by showing that a proof of [7] can be tailored to suit the (j, m) -scheme [6], from which one can also extract the m -scheme [5], that gives us a one-to-one map from the flux quantum numbers to m . Before we go into the technical details, let us choose units such that $4\pi\gamma\ell_P^2 = 1$, where γ is the Barbero-Immirzi parameter involved in the quantization and ℓ_P the Planck length. In these units, counting is possible only for integral classical areas

$$A_{class} = k, \quad (1)$$

where k is a positive integer labelling the Chern-Simons theory on the horizon. This poses no serious problem in the semiclassical limit, since assuming $\gamma \sim o(1)$, the pre-factor $4\pi\ell_P^2 \sim 10^{-69} \text{ m}^2$, which is an extremely small number. So the area intervals ΔA for which no counting is possible are also extremely small, which essentially gives a continuum of classical areas. The area eigenvalue for a given configuration of spins is

$$A_{N[j,m]} = 2 \sum_{j,m} N[j,m] \sqrt{j(j+1)} \quad (2)$$

where $N[j, m]$ is the number of punctures on the horizon carrying spin quantum numbers (j, m) .

The *horizon* states are labelled by the flux quantum numbers b on the punctures; these are elements of \mathbb{Z}_k obeying the restriction

$$\sum_b N_b b = 0 \text{ mod } k \quad (3)$$

where N_b is the number of punctures carrying the flux quantum number b . The quantum isolated horizon boundary conditions imply that the flux quantum number b associated with each puncture must be related to the spin projection quantum number m associated with that puncture as

$$b = -2m \text{ mod } k. \quad (4)$$

Then by (3), the spin projections obey the constraint

$$\sum_{j,m} N[j, m] m = \frac{nk}{2} \quad (5)$$

where $n \in \mathbb{N}$ is some integer. This constraint will now be sharpened, as in [7], but in our analysis we shall keep the j instead of replacing them through inequalities by m so that the choice between the two kinds of counting is left open.

The area eigenvalues of interest are taken in a range

$$k - \epsilon \leq A_{N[j,m]} \leq k + \epsilon \quad (6)$$

where $\epsilon < k$ is a positive number, a macroscopic parameter that should be independent of the microscopic configurations which are summed over in calculating the total number of microstates. Now

$$\begin{aligned} A_{N[j,m]} &\geq 2 \sum_{j,m} N[j,m] \sqrt{|m|(|m|+1)} \quad \text{since } j \geq |m| \\ &\geq 2 \sum_{j,m} N[j,m] |m| + \sum_{j,m} N[j,m] (\sqrt{3}-1) \end{aligned} \quad (7)$$

where in the last step we have made use of the fact that the quantity $[|m|(|m|+1)]^{1/2} - |m|$ increases monotonically with $|m|$ and $(\sqrt{3}-1)/2$ is its lowest value, obtained from $m = 1/2$, $m = 0$ being unphysical because of the boundary condition (4) which says that for $m = 0$ the flux quantum number $b = 0 \bmod k$ and such punctures are invisible in Chern-Simons theory. Now by (5) the series

$$\sum_{j,m} N[j,m] |m| \geq \left| \sum_{j,m} N[j,m] m \right| = \frac{|n|k}{2}. \quad (8)$$

Thus since $k + \epsilon \geq A_{N[j,m]}$, from (7) and (8) we get

$$k + \epsilon \geq |n|k + \sum_{j,m} N[j,m] (\sqrt{3}-1). \quad (9)$$

So for all nonzero values of $|n|$ either we cannot choose ϵ irrespective of the microscopic configurations (for $|n| = 1$) or ϵ becomes larger than k (for $|n| > 1$) since $\sum_{j,m} N[j,m] > 1$. The only allowed value is $n = 0$. So the *spin-projection constraint* takes a more restricted form

$$\sum_{j,m} N[j,m] m = 0. \quad (10)$$

The sum $\sum N[j,m] (\sqrt{3}-1) = N(\sqrt{3}-1)$, where N is the total number of punctures, is a large number $\sim o(k)$ for a large black hole. Hence $\epsilon < N(\sqrt{3}-1)$. From (7) we get $2 \sum N[j,m] |m| \leq k + [\epsilon - N(\sqrt{3}-1)] < k$. So for a large black hole ($k \gg 1$) we get a further restriction

$$\sum_{j,m} N[j,m] |m| < \frac{k}{2}. \quad (11)$$

If one is interested in small values of k , *i.e.* one attempts to accommodate small black holes, one must take $\epsilon < \sqrt{3}-1$ to achieve (11) because N is not a large number but > 1 for a nontrivial Chern-Simons theory.

Naïvely, the inequality (11) implies that each $|m| < k/2$; however, the most stringent upper bound comes from the configuration with minimum N , that is $N = 2$. In view of the fact that the two m must sum to zero, it follows that the upper bound is $|m| < k/4$, which also holds for cases with more than two punctures.

Now given a flux quantum number b in general there are many spin quantum numbers $m = -b/2 + nk/2$ where $n \in \mathbb{N}$ is some integer which could be zero. At most one of these m can be less than $k/4$ in magnitude. To see this, suppose the minimum of $|-b/2 + nk/2|$ is at $n = n_0$ and $|-b/2 + n_0 k/2| < k/4$. Then for integer p , $|p| \geq 1$ the other values $n_0 + p$ will not satisfy the bound since $|-b/2 + (n_0 + p)k/2| \geq |p|k/2 - |-b/2 + n_0 k/2| > (2|p|-1)k/4 \geq k/4$. But this proof relies on the assumption that there exists an integer n_0 for which $|-b/2 + n_0 k/2| < k/4$. This assumption breaks down for the special values $b = \pm k/2$, in which case the condition requires $|2n_0 \pm 1| < 1$ for which there is no solution for n_0 , implying that such a b has no corresponding m .

Thus, only if the domain of the surface states is restricted to exclude the values $b = \pm k/2$ for flux quantum numbers does the map from flux quantum numbers to the associated spin quantum numbers become one-to-one. Now if one takes the point of view that only surface states represent true black hole microstates, then one can use this one-to-one map to count the bulk states carrying only the m quantum numbers. As the bulk states are characterized by both j, m , clearly one must consider as equivalent the states that have different j but same m . Note that since the constraints (6) and (10) involve the bulk quantum numbers j, m , it is essential to consider the bulk states in order to take care of the constraints. In fact, the above reasoning illustrates the fact that the equivalence of b and m quantum numbers has nothing to do with how one characterizes states in the effective theory, *i.e.* whether or not one should regard j quantum numbers as relevant. This leaves us at this stage with the choices of (j, m) and m -counting. We shall now argue in favour of one.

In general, a complete list of quantum numbers to label the states of a black hole or of an isolated horizon remains an open issue in loop quantum gravity because a complete set of observables is yet to be constructed in the full

theory. In fact, we expect a full quantum theory of black holes not to be a theory on the surface alone (unless some sort of holography is at work, which is quite unlikely in loop quantum gravity). In an effective case, one assumes that only a partial set of observables is relevant – the others being ‘slow’ and not so relevant for the purpose of describing static or equilibrium properties of a black hole, contributing only in its dynamical properties – namely the area and the flux. While the former is associated with the bulk Hilbert space, the latter is associated with the surface Hilbert space. Boundary conditions provide a one-to-one mapping between the bulk quantum number m and the flux quantum number b in the sense explained above. The additional j labels, which determine the area, are irrelevant so far as the surface states are concerned and this is the reason why the m -scheme was conceived, but the idea that the Chern-Simons theory represents the effective theory of the horizon is misleading. An effective theory of the horizon is not necessarily a theory *on* the horizon. A quantum horizon is characterized by a set of flux quantum numbers, an area eigenvalue and a number of boundary conditions. Only the first one of these is implementable on the surface states alone, hence any surface theory is inadequate to define a quantum isolated horizon. It is clear that an effective theory of the horizon must involve both a surface theory and a bulk theory. The bulk degrees of freedom provide – at least in the description that is at hand [3] – a relevant observable for the horizon, the area, which is kept *fixed*. Different eigenstates must be used as physically distinct quantum states and thus the (j, m) -scheme is needed. In our view a genuine field theory of an isolated horizon must be described by the Hilbert space $\mathcal{H}_v \otimes \mathcal{H}_s$ and hence the surface labels are inadequate to characterize a quantum isolated horizon.

Illustration of state counting: To illustrate the details of the precise counting procedures, we consider a small black hole with $A = 4\sqrt{6} \approx 9.80$. The *exact* eigenvalue corresponds to 2 punctures with $j = 2, 2$. Each puncture in principle has 5 allowed values for m , but not all the 25 states obey (11), which is satisfied only if $m = \pm 2, \mp 2$, $m = \pm 1, \mp 1$ or $m = 0, 0$, so that there are 5 states satisfying those conditions. However, as indicated above, the m quantum number has to be non-zero, so there are only 4 states corresponding to this area in m counting. This is also the number of states in (j, m) counting.

Instead of fixing the exact eigenvalue at $A = 4\sqrt{6}$, we may fix the classical area at $k = 10$ and count states with nearby eigenvalues. Possibilities with 2 punctures are $j = 2, 2(m = \pm 2, \mp 2 \text{ or } m = \pm 1, \mp 1)$, $j = \frac{5}{2}, \frac{3}{2}(m = \pm \frac{3}{2}, \mp \frac{3}{2} \text{ or } m = \pm \frac{1}{2}, \mp \frac{1}{2})$, $j = 3, 1(m = \pm 1, \mp 1)$ and $j = \frac{7}{2}, \frac{1}{2}(m = \pm \frac{1}{2}, \mp \frac{1}{2})$.

Possibilities with 3 punctures are $j = 3, \frac{1}{2}, \frac{1}{2}(m = \pm 1, \mp \frac{1}{2}, \mp \frac{1}{2})$, $j = \frac{5}{2}, 1, \frac{1}{2}(m = \pm \frac{3}{2}, \mp 1, \mp \frac{1}{2} \text{ or } m = \pm \frac{1}{2}, \mp 1, \pm \frac{1}{2})$, $j = 2, 1, 1(m = \pm 2, \mp 1, \mp 1)$ and $j = 2, \frac{3}{2}, \frac{1}{2}(m = \pm 2, \mp \frac{3}{2}, \mp \frac{1}{2} \text{ or } m = \pm 1, \mp \frac{3}{2}, \pm \frac{1}{2})$. There are other possibilities as well.

In m counting, $j = 2, 2(m = \pm 1, \mp 1)$ and $j = 3, 1(m = \pm 1, \mp 1)$ are not distinguished, just as $j = \frac{5}{2}, \frac{3}{2}(m = \pm \frac{1}{2}, \mp \frac{1}{2})$ and $j = \frac{7}{2}, \frac{1}{2}(m = \pm \frac{1}{2}, \mp \frac{1}{2})$ are not. However, these states correspond to distinct areas and (j, m) counting recognizes them as different. Similarly $j = 3, \frac{1}{2}, \frac{1}{2}(m = \pm 1, \mp \frac{1}{2}, \mp \frac{1}{2})$ and $j = 1, \frac{5}{2}, \frac{1}{2}(m = \pm 1, \mp \frac{1}{2}, \mp \frac{1}{2})$ have different areas but are treated as same in m counting, just as $j = \frac{5}{2}, 1, \frac{1}{2}(m = \pm \frac{3}{2}, \mp 1, \mp \frac{1}{2})$ and $j = \frac{3}{2}, 2, \frac{1}{2}(m = \pm \frac{3}{2}, \mp 1, \mp \frac{1}{2})$ are. Thus the number of states is less in this prescription.

It may be noted that each $|m| \leq 2 < \frac{5}{2}$ here, as expected for this case.

An $|m| = j$ rule? Very few of the above states in m counting satisfy the $|m| = j$ rule. To understand the motivation for setting $|m| = j$, we may recall a theorem from [7]. In m counting, the number of states with $2 \sum \sqrt{j(j+1)} \leq A$ is equal to the number of states with $2 \sum \sqrt{|m|(|m|+1)} \leq A$. This is useful in the counting of states for large black holes, where one can consider the range $2 \sum \sqrt{j(j+1)} \leq A$ instead of $A - \epsilon \leq 2 \sum \sqrt{j(j+1)} \leq A + \epsilon$. Then one can count *only the states with* $|m| = j$. But this can be done *only if large areas are considered*, otherwise the inequality $2 \sum \sqrt{j(j+1)} \leq A$ is not appropriate.

Counting with $SU(2)$: Let us now discuss the suggested use of $SU(2)$ Chern-Simons theory as the effective quantum field theory of isolated horizons [10]. Two representations of $SU(2)$ are involved here; one is associated with the bulk Hilbert space and the other with the surface Hilbert space. The surface spin quantum numbers (j_s, m_s) will be related to the bulk labels (j, m) via the boundary conditions only. In this case the holonomy matrices $h \in SU(2)$ around each puncture acting on the surface states is to be matched with the exponentiated triad e restricted to the edge that is attached to the puncture acting on the bulk states. Since both the operators are $SU(2)$ group elements, the matching condition $h\Psi_v \otimes \mathbb{I}\Psi_s = \mathbb{I}\Psi_v \otimes e\Psi_s$ can be implemented strongly only when both Ψ_s, Ψ_v are respectively eigenstates of h, e with equal eigenvalues. The result will be a condition like $m_s = -m$ for each puncture, similar to (4), together with $j_s = j$, which arises because the dimensionalities of the bulk and the surface representations have to match. However, there is another consistency condition on the surface states: the product of all holonomies must be the identity, which amounts to the singlet condition that the total spin $\sum \mathbf{j}_s$ vanishes on states. As $j_s = j$, one can rewrite the area eigenvalue in terms of surface labels, unlike the $U(1)$ case. The singlet condition even takes care of the $\sum m_s = 0$ condition more strongly. One can count $SU(2)$ Chern-Simons states taking care of these constraints. The calculation is similar to the (j, m) procedure in the $U(1)$ theory, and most of the above relations hold here, with minor changes. One difference is that $m = 0$ is not to be excluded here, but only $j = 0$, because only the latter implies

a trivial puncture. Another difference is in the relation between the level k and the classical area, which here is

$$A_{class} = \frac{1 - \gamma^2}{2} k. \quad (12)$$

With the appropriate factor in front of k , the earlier restrictions on m hold once again. Of course, the number of states will be reduced here by the $SU(2)$ singlet condition.

As an illustration, we may consider the exact area eigenvalue $A = 4\sqrt{6}$ again, so that there are two punctures with $j = 2$ on each. There is only one singlet state that can be constructed out of two such spins.

Conclusion: In summary, if one chooses to count $U(1)$ states with distinct m -labels [7] instead of states with distinct j, m -labels [6], one fails to distinguish between some states which have different areas. The mapping between the allowed values of b and m is one-to-one, but not all values of b are allowed. As regards the $|m| = j$ prescription, it does not even count all states with distinct m -labels: it gives a reduced value except in special cases involving $j = \frac{1}{2}$ or for large area [4, 5]. The $SU(2)$ formulation is analogous to the j, m procedure and simpler as far as counting is concerned.

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